How interactions resolve state-dependence in a holographic toy model for black holes

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based on work with A.Bzowski, T. Hertog
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Outline

Black holes

Black holes as a Quantum Gravity lab

- Address problems requiring an extension of GR
- String Theory provides a UV completion, in a unified theory, of GR and QFT
- Exploit AdS/CFT as a tool to investigate quantum gravity

Main areas of investigation

1. Microscopic explanation of black hole entropy
2. Formulation of information paradox in terms of entanglement of quantum fields

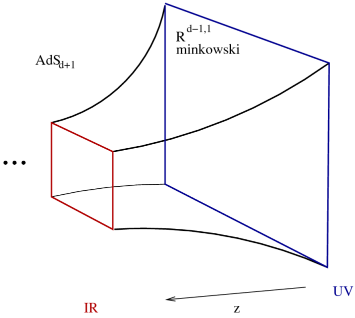


Fig. from McGreevy, 2009

Need for a toy model

Information paradox = incompatibility among: [AMPS, '13]

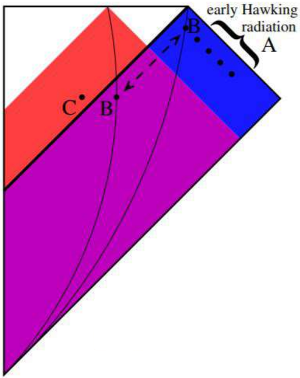
- Unitary black hole formation and evaporation
- Validity of the semiclassical approximation for an asymptotic observer
- Existence of black hole microstates visible to an asymptotic observer as states with exponentially small energy differences, thus $S_{BH} = \log(\#\mathrm{dof})$
- An infalling observer in the near horizon region experiences the vacuum

From AdS/CFT : unitarity is preserved
Black hole inner region is cut out from the physical universe (firewall), OR effective field theory is not appropriate to describe non perturbative quantum gravity.

How to characterize the breakdown of EFT?

$\rightarrow$ Study the problem in a toy model

Entanglement of fields in a black hole background



pic. from R. Bousso

- Hawking radiation, '76 - near horizon region is Rindler, the Hilbert space decomposes as $\mathcal{H} = \mathcal{H}_L \otimes H_R$ (inside and outside the horizon)
- Mathur's theorem '08: Violation of Strong Subadditivity Principle for entanglement entropy S_E of Hawking radiation.
- In a dual picture, from AdS/CFT, unitarity of black hole evaporation is guaranteed [Kaplan at al. '17]

Proposals

1. *Firewalls:* Give up the Equivalence principle: the near horizon region is not vacuum, the inner region is not accessible, it is cut out from spacetime [Almheiri, Marolf, Polchinski, Sully '13]
2. *State Dependence:* Evade strong subadditivity principle by identifying modes outside the horizon with modes inside the horizon through *mirror* operators [Papadodimas, Raju '13]. Truncation of the operator algebra, problems of overcompleteness of the basis (Jafferis).
3. *Vacuum structure:* Mathur's theorem: quantum gravity effects cannot prevent information loss if (they are confined to within a given scale and) if the vacuum of the theory is assumed to be unique. [Hawking, Strominger, Perry '15]
4. Non-perturbative effects play a fundamental role (Giddings), factorization of Hilbert space break down for semiclassical gravity (Harlow-Jafferis)...

A toy model

Black hole paradoxes

- Reformulation of the information paradox in terms of purely field theoretic data
- Investigate the nature of non-perturbative quantum gravity corrections to black hole physics.

Toy model

→ A new holographic tow model that incorporates a specific proposal for the nature of QG corrections to BH physics

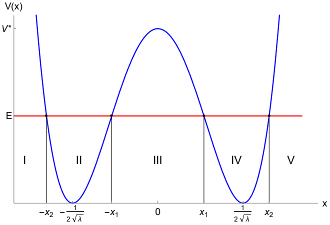
→ The model consists of a quantum mechanical particle in a double well potential.

Double well potential

$$V(x) = \frac{1}{32\lambda} - \frac{1}{4}\omega^2 x^2 + \frac{\lambda}{2}\omega^4 x^4$$

governed by the standard Hamiltonian

$$H = \frac{1}{2}p^2 + V(x)$$



- Semiclassical vacuum states: φ_0^L, φ_0^R, around the minima

$$x_{L,R} = \frac{1}{2\omega\sqrt{\lambda}} \qquad \rightarrow \qquad \lambda \sim \frac{1}{N^2}$$

- Max at $V_* = \frac{1}{32\lambda}$
- Define a *decoupling* limit as $\lambda \rightarrow 0$. Physically, the system can be thought of two decoupled harmonic oscillators, described by a tensor product Hilbert space $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$.

Goal of the toy model

- Construct a well defined perturbative theory in the regime $\lambda \ll 1$, as the tensor product of two harmonic oscillators

$$\mathcal{H}^{eff} = \mathcal{H}_L \otimes \mathcal{H}_R$$

- This effective field theory captures many paradoxes associated with the semiclassical treatment of black holes
- Explore dynamical processes, for example tunneling and evolution of coherent states
- Quantify how interactions resolve the paradoxes

caveat: will work with fix frequency $\omega = 1$, will not be able to capture thermal behavior, for which an ensemble of modes is required.

Hairy black holes in N=8 Supergravity

Effective $\mathcal{N} = 8$, 4D gauged supergravity from M theory on $AdS_4 \times S^7$

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}(\nabla\phi)^2 + 2 + \cosh(\sqrt{2}\phi)\right]$$

AdS_4 vacuum ($\ell^2 = 1$) at $\phi = 0$
Scalar masses $m^2 = -2$, the scalar ϕ of an asymptotically AdS_4 solution decays at large radius as

$$\phi(r) \sim \frac{\alpha}{r} + \frac{\beta}{r^2}\ , \qquad r \to \infty$$

The dual, boundary theory is defined by the choice of boundary condition $\beta(\alpha)$, corresponding to multi trace boundary terms

$$W(\alpha) = \int_0^\alpha \beta(\tilde{\alpha})d\tilde{\alpha}\ ,$$

defines *designer gravity* boundary conditions: $\int W(\mathcal{O})$ perturbation to the dual CFT action.
Certain deformations W give rise to field theories with additional vacua.

Hairy black holes in N=8 supergravity

[Hertog, Horowitz, '04]

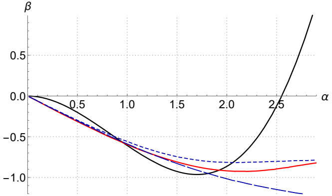
Conserved mass of spherical solutions is given by

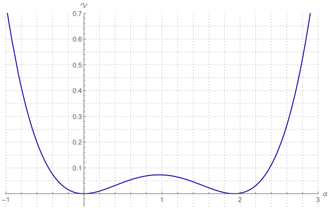
$$M = \mathrm{Vol}(S^2)\,[M_0 + \alpha\beta + W]$$

The precise correspondence between solitons and field theory vacua is given by

$$\mathcal{V}(\alpha) = -\int_0^\alpha \beta_s(\tilde{\alpha})d\tilde{\alpha} + W(\alpha)$$

where $\beta_s(\alpha)$ is obtained from the asymptotic scalar profiles of spherical soliton solutions with different values $\phi(0)$ at the origin $r = 0$.

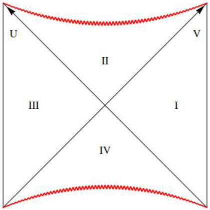


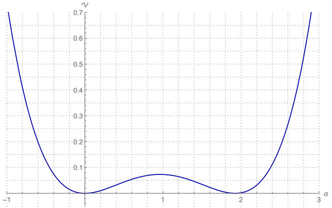


Toy model interpretation

1. Single-sided black hole solutions in global AdS with scalar hair.
   - Perturbative degrees of freedom on both sides of the horizon, coupled through multi-trace interactions.
   - Decoupling limit corresponds to singular horizon.
2. Not only entanglement but also interactions between the boundary CFTs.
   ⋆ Perturbative vacua dual to the two asymptotic regions on both sides of the horizon.

→ The potential barrier in the dual toy-model is a proposal for a specific interaction between left and right modes in the bulk, $|n_k\rangle_L$ and $|n_k\rangle_R$

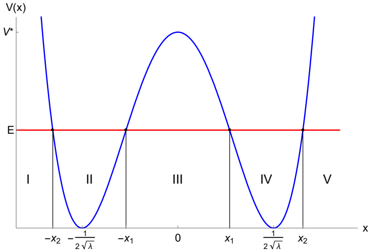




Double well potential

$\rightarrow$ Canonical quantization either around x_L or around x_R, ignoring all interaction terms.

$\rightarrow$ Two separate Fock spaces $\mathcal{F}_L$ and $\mathcal{F}_R$ equipped with two pairs of creation and annihilation operators b_L, b_L^+ and b_R, b_R^+.



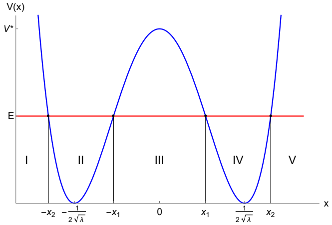
However, while formally independent, these two Fock spaces must be related since they arise from the same system.

The theory around semiclassical vacua

Perturbation theory $H = H_R^{(0)} + H_R^{(1)}$,
$y_R = x - x_R$

$$H_R^{(0)} = \frac{1}{2}p^2 + \frac{1}{2}y_R^2, \qquad H_R^{(1)} = \sqrt{\lambda}y_R^3 + \frac{\lambda}{2}y_R^4,$$

Standard canonical quantization based on $H_R^{(0)}$ around x_R yields a Fock space $\mathcal{F}_R$ with a set of basis states $|n\rangle_R$, together with a pair of creation-annihilation operators b_R, b_R^+



$$b_R|0\rangle_R = 0, \qquad |n\rangle_R = \frac{1}{\sqrt{n!}}(b_R^+)^n|0\rangle_R \ .$$

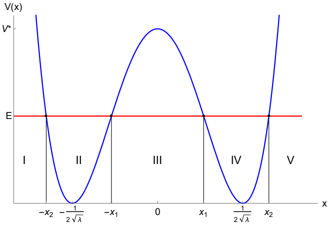
- Associate to these Fock spaces two observers, obs_L and obs_R
- The state $|0\rangle_R$ as the natural semiclassical vacuum for obs_R, the excited state $|n\rangle_R$ is n-particle state.
- The left observer regards $|0\rangle_L$ as the semiclassical vacuum and $|n\rangle_L$ as an n-particle state.

The theory around semiclassical vacua

HO normalized eigenfunctions $\{\varphi_n\}_{n\in\mathbb{N}}$

$$\varphi_n(x) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} H_n(x) e^{-\frac{x^2}{2}} \,, \qquad x \in \mathbb{R} \,.$$

Cannot compute $[b_L, b_R]$ as the operators act on different Hilbert spaces.
$\rightarrow$ Relate $\mathcal{F}_L$ and $\mathcal{F}_R$



$$F_R \;:\; \mathcal{F}_R \ni |n_R\rangle \mapsto \varphi_n^R \in \mathcal{H}, \qquad \varphi_n^R(x) = \varphi_n(x - x_R) \,,$$
$$F_L \;:\; \mathcal{F}_L \ni |n_L\rangle \mapsto \varphi_n^L \in \mathcal{H}, \qquad \varphi_n^L(x) = (\Theta\varphi_n^R)(x) = (-1)^n \varphi_n(x - x_L) \,.$$

CPT operator Θ acts on elements $\psi \in \mathcal{H}$ as $(\Theta\psi)(x) = \psi^*(-x)$
Total Hilbert space $\mathcal{H}$ is isomorphic to each Fock space $\mathcal{F}_L$ and $\mathcal{F}_R$ *separately*,

$$\mathcal{H} \cong \mathcal{F}_R \cong \mathcal{F}_L \;;$$

There is no tensor product. The interactions provide a non-trivial identification of the two Fock spaces within a single $\mathcal{H}$.

The theory around semiclassical vacua

Compare the perturbative description of the asymptotic observers.

- Define new annihilation operators a_R, a_L constructed from b_R, b_L that do have a well defined action in $\mathcal{H}$:

$$a_R = F_R b_R F_R^{-1}, \qquad a_L = F_L b_L F_L^{-1},$$

- Their action is

$$a_R \varphi_n^R = \sqrt{n} \varphi_{n-1}^R, \qquad a_R^+ \varphi_n^R = \sqrt{n+1} \varphi_{n+1}^R \ ,$$
$$a_L \varphi_n^L = \sqrt{n} \varphi_{n-1}^L, \qquad a_L^+ \varphi_n^L = \sqrt{n+1} \varphi_{n+1}^L \ ,$$

- The action of a_L is related to the action of a_R by the parity operator,

$$a_L = \Theta a_R \Theta$$

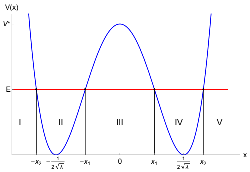
Notice: Left and right creation-annihilation operators on one side and the other of the horizon are related in Papadodimas-Raju proposal.

Perturbative theory operators?

Creation-annihilation operators are related as

$$a_L = -\frac{1}{\sqrt{2\lambda}}\mathbb{I} - a_R = -\frac{N}{\sqrt{2}}\mathbb{I} - a_R.$$

since the eigenfunctions φ_L, φ_R are related by a displacement.



- It does not possess a finite decoupling limit $N \to \infty$ as an operator statement: instead of approximating the free field creation-annihilation operators b_L, b_L^+, b_R, b_R^+, these operators diverge in the decoupling limit.
- These operators have wrong commutation relations

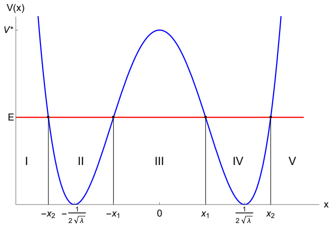
$$[a_L, a_R] = [a_L^+, a_R^+] = 0, \qquad [a_L, a_R^+] = [a_R, a_L^+] = -1 \ .$$

in sharp contrast with the expectation of black hole physics that left and right creation-annihilation operators commute as a consequence of the locality of semiclassical physics near the horizon.

Decoupling limit

- $\lambda \to 0$ limit requires some care
- We expect to end up with two separate harmonic oscillators with the tensor product Hilbert space $\mathcal{H}_0 \cong \mathcal{F}_L \otimes \mathcal{F}_R$
- However, the interaction Hamiltonian $H_R^{(1)}$ vanishes for $\lambda = 0$:

$$H_R^{(1)} = \sqrt{\lambda} y_R^3 + \frac{\lambda}{2} y_R^4$$

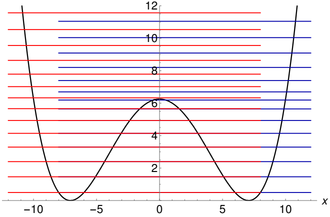


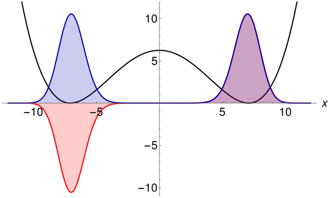
From the point of view of the vacuum at x_R, the second vacuum moves away and a single harmonic oscillator Hilbert space $\mathcal{F}_R$ remains, it looks like every state φ_n^L disappears as $\lambda \to 0$, so the limit is singular.

Notice: This is also the case in the bulk for the black holes with scalar hair - In the limit in which the vacua in the dual theory decouple, a curvature singularity at the horizon develops, effectively dividing the inside and outside regions in two separate spacetimes.

Energy levels

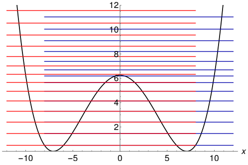
Energy eigenstates and energies can be computed numerically to arbitrary precision

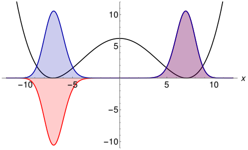




For every quantity in the effective theory regime:
$f(\lambda)$ is *non-perturbatively small* if $f \sim 0$ as $\lambda \to 0^+$, where $\sim$ denotes the asymptotic expansion.
$f(\lambda)$ is non-perturbatively small if $f(\lambda) = o(\lambda^n)$ for all $n \geq 0$ as $\lambda \to 0^+$
(non-perturbatively small terms $\sim o(\lambda^\infty)$)

Energy levels





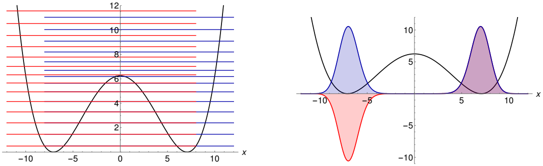
Double well potential is invariant under $x \to -x$, $[\Theta, H] = 0$.
Decompose the Hilbert space as $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\Theta\mathcal{H}^+ = \mathcal{H}^+$ and $\Theta H^- = -\mathcal{H}^-$. We denote energy eigenstates by

$$H\Psi_n^{\pm} = E_n^{\pm}\Psi_n^{\pm} \ ,$$

The corresponding energies satisfy $E_n^+ < E_n^-$ and their difference $\Delta E_n = E_n^- - E_n^+$ is exponentially small (instanton effect)

$$E_0^- - E_0^+ = \frac{2}{\sqrt{\pi\lambda}} e^{-\frac{1}{6\lambda}} \left[1 + O(\lambda)\right].$$

Microstates



$\Delta E_n \sim 0$ in general is a non-perturbative effect, hence exponentially small for energies $E_n^{\pm} < V_* = 1/(32\lambda)$.

- **Interpr:** every pair of energy eigenstates $\Psi_n^{\pm}$ corresponds to two microstates with exponentially small energy splitting due to non-perturbative effects.
- **Dual description of black holes in N=8 SG**:
  - (perturbative) dof on both sides of the horizon $\rightarrow$ excitations around different perturbative vacua: significant support around both minima of the potential as the dual description of a *black hole microstate*
  - semiclassical states centered around one of the two vacua only $\rightarrow$ spacetimes without a black hole

Microstates

There are microstates of any energy $E \ll V_*$. Consider the lowest energy states and the space

$$\mathcal{M} = \{\alpha_+ \Psi_0^+ + \alpha_- \Psi_0^- \ : \ \alpha_\pm \in \mathbb{C}\}.$$

for any normalized state $\mu \in \mathcal{M}$, its energy is non-perturbatively close to the ground state energy

$$\langle \mu | H | \mu \rangle = E_0^+ + o(\lambda^\infty)$$

In perturbation theory the ground state is degenerate

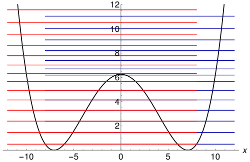
- $\mu \in \mathcal{M}$ is the subspace of *perturbative vacua*
- Each element $\mu \in \mathcal{M}$ is a *perturbative vacuum*

$\sim$ a correspondence between the (degenerate) energy E_n of the states and the mass of the black holes.

Microstates

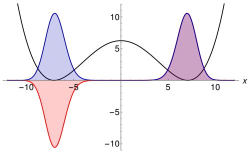
→ Single asymptotic observer with easy access to the right portion of the wave function only

- The right portion of the wave function specifies a macrostate
- Microstates differ in the shape of the wave function around x_L



→ A pair of observers in two distinct asymptotic regions as *e.g.*, in the case of double sided, eternal black hole.

- A macrostate is given by two independent pieces of the wave function: the left portion, ψ_L, and the right portion, ψ_R represented by a tensor product $\psi_L \otimes \psi_R$ - microstates are all states of the form $\alpha_L \psi_L + \alpha_R \psi_R$ for $\alpha_L, \alpha_R \in \mathcal{C}$.
- Each microstate is a specific continuation through the potential barrier that eludes both observers.



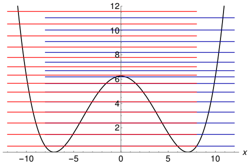
Microstates

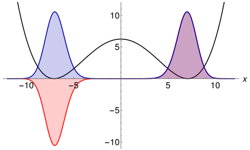
→ In perturbation theory various microstates cannot be distinguished:

$$S_B = \log \dim \mathcal{H}_{\text{fine}} = \log 2.$$

Bolzmann entropy associated with a single pair of harmonic oscillators

There is no *ensemble*





Firewalls

Number operators for L,R asymptotic observers

$$N_L = H_L^{(0)} = a_L^+ a_L, \qquad N_R = H_R^{(0)} = a_R^+ a_R$$

An infalling observer has access to perturbative physics inside and outside the horizon:

$$N_A = N_L + N_R + O(\sqrt{\lambda}) = a_L^+ a_L + a_R^+ a_R + O(\sqrt{\lambda}),$$

- at $\lambda = 0$: sum of excitations of two decoupled harmonic oscillators
- at small coupling: corrections of order $O(\sqrt{\lambda})$?
  - For semiclassical vacua

$$\langle \varphi_0^L | N_L | \varphi_0^L \rangle = \langle \varphi_0^R | N_R | \varphi_0^R \rangle = 0 \ , \quad \langle \varphi_0^L | N_R | \varphi_0^L \rangle = \langle \varphi_0^R | N_L | \varphi_0^R \rangle = \frac{1}{2} N^2$$

  - For typical states

$$\langle \psi | N_A | \psi \rangle \gtrsim \frac{1}{2} N^2 \left[1 + O\left(\frac{1}{N} \right) \right] .$$

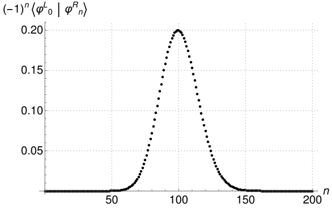
Firewalls

- Energy of the states remains small:

$$\langle \varphi_0^L | H | \varphi_0^L \rangle = \langle \varphi_0^R | H | \varphi_0^R \rangle = \frac{1}{2} + \frac{3}{8}\lambda.$$

- For obs_R, the semiclassical vacuum state φ_0^L is a highly excited state.

$$\langle \varphi_0^L | \varphi_n^R \rangle = \frac{(-1)^n e^{-\frac{1}{4\lambda}}}{\sqrt{2^n \lambda^n n!}}$$



For obs_R, highly energetic modes are excited with max: $n \sim \frac{1}{2\lambda} \sim N^2$.

Firewalls

Relation with the ground state of the full Hamiltonian:

$$\Psi_0^{\pm} = \frac{1}{\sqrt{2}}(\varphi_0^R \pm \varphi_0^L) + O(\sqrt{\lambda}).$$

States of low energy, highly populated both with respect to N_L and N_R and

$$\langle \Psi_0^{\pm} | N_R | \Psi_0^{\pm} \rangle = \langle \Psi_0^{\pm} | N_L | \Psi_0^{\pm} \rangle = \frac{1}{4} N^2 \left[1 + O\left(\frac{1}{N}\right) \right].$$

$\rightarrow$ the number operator N_A, on generic microstate of the form $\mu \in \mathcal{M}$, shows a firewall

- Microstates are indistinguishable by $obs_{L,R}$, with access to the perturbative physics only.
- Interactions: non-perturbative level splitting $\rightarrow$ source for entropy.

Perturbation theory

How to understand what happens?

- From the interacting model, identify the perturbative degrees of freedom for $obs_{L,R}$

$$\hat{\psi}_n^{\pm} \qquad \in \qquad \mathcal{H}_{pert}$$

- Construct well-defined, perturbative creation annihilation operators

$$\hat{a}_{L,R}, \hat{a}^{\dagger}_{L,R} \qquad on \qquad \mathcal{H}$$

- Verify they have a well defined decoupling limit
- Identify a tensor product structure for the effective theory
- Connect the effective theory to the full theory (operators, states..)

Limits of perturbation theory

Perturbation theory breaks down around $x_{L,R}$, when the occupacy numbers are $n, m \sim N^2 \equiv \frac{1}{\lambda}$

- The overlap between the left and right semiclassical modes becomes significant:

$$\langle \varphi_n^L | \varphi_n^R \rangle = \frac{e^{-\frac{1}{4\lambda}}}{(2\lambda)^n n!} \left[1 + O(\lambda)\right].$$

- When n is of order $1/\lambda$ the correction is of the same order than the unperturbed part

$$\langle \varphi_n^R | H | \varphi_n^R \rangle = \frac{1}{2} + n + \frac{3}{8}\lambda(2n^2 + 2n + 1).$$

- after a time $t \sim N$

$$\langle \varphi_n^R | e^{-i\,tH} | \varphi_m^R \rangle = \delta_{nm} - i\,t \langle \varphi_n^R | H_R^{(1)} | \varphi_m^R \rangle + \ldots$$

- The subleading terms in the commutation relation

$$[H, a_R] = -a_R - \frac{3}{\sqrt{2}}\sqrt{\lambda} y_R^2 - \lambda\sqrt{2} y_R^3$$

become relevant whenever $n \sim N^2$.

Summary: deconstruction of DW potential

- Identification of the notion of
  - Asymptotic Observer
  - Perturbative Vacua
  - Semiclassical States and their Fock space
  - "dual black hole" microstates
- Perturbation theory cannot resolve the fine-grained feature of microstates

  $\rightarrow$ Non-perturbative level splitting is a source of *entropy*.
- Creation-annihilation operators do not possess a well defined $\lambda \rightarrow 0$ limit

  $\rightarrow$ They cannot be used to define creation-annihilation operators for the asymptotic region.

To do: correctly identify perturbative degrees of freedom as perceived by asymptotic observers.

Effective theory

Is it possible to define:

$$\hat{a}_R \varphi_n^R \stackrel{?}{=} \sqrt{n}\varphi_{n-1}^R, \qquad \hat{a}_R \varphi_n^L \stackrel{?}{=} 0,$$

$$\hat{a}_R^+ \varphi_n^R \stackrel{?}{=} \sqrt{n+1}\varphi_{n+1}^R, \qquad \hat{a}_R^+ \varphi_n^L \stackrel{?}{=} 0$$

A new number operator $\hat{N}_A$ defined by means of the hatted operators

$$\hat{N}_A = \hat{a}_L^+ \hat{a}_L + \hat{a}_R^+ \hat{a}_R$$

would then act on the energy eigenstates $\Psi_0^\pm$

$$\hat{N}_A |\Psi_0^\pm\rangle = \frac{1}{\sqrt{2}}(a_R^+ a_R \varphi_0^R \pm a_L^+ a_L \varphi_0^L) + O(\sqrt{\lambda}) = 0 + O(\sqrt{\lambda}).$$

$\rightarrow$ no firewall

PB Such operator does not exist, the basis $\{\varphi_m^L, \varphi_n^R\}_{n,m}$ is an overcomplete basis

1. truncate the basis at finite $n \leq \bar{N}$
2. projected operators

Effective theory

Proposal for 'orthogonalizing' the overcomplete basis $\{\varphi_n^L, \varphi_n^R\}_n$

- symmetric and antisymmetric combinations of all energy eigenstates,

$$\Psi_n^L = \frac{1}{\sqrt{2}}(\Psi_n^+ - \Psi_n^-), \qquad \Psi_n^R = \frac{1}{\sqrt{2}}(\Psi_n^+ + \Psi_n^-)$$

  two Hilbert subspaces

$$\mathcal{H}_L = \text{span}\{\Psi_n^L\}_n, \qquad \mathcal{H}_R = \text{span}\{\Psi_n^R\}_n.$$

- $\langle \Psi_m^L | \Psi_n^R \rangle = 0$

$$\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R, \qquad \mathcal{H}_L \perp \mathcal{H}_R, \qquad \Theta\mathcal{H}_L = \mathcal{H}_R, \qquad \Theta\mathcal{H}_R = \mathcal{H}_L.$$

- Projected operators : $\hat{a}_L = P_L a_L P_L, \qquad \hat{a}_R = P_R a_R P_R.$
- Number operator on the direct product space is

$$\hat{N}_A = P_L N_L P_L \oplus P_R N_R P_R = P_L a_L^+ a_L P_L + P_R a_R^+ a_R P_R.$$

- Ψ_m^L, Ψ_n^R are perturbative states

Perturbative Hilbert spaces

Define $\hat{N}^a$ on perturbative states $\psi = \alpha_L \varphi_m^L + \alpha_R \varphi_n^R$, on which:

$$\langle \psi | \hat{N}_A | \psi \rangle = |\alpha_L|^2 m + |\alpha_R|^2 n + O(\sqrt{\lambda}).$$

- $\Psi_n^R = \varphi_n^R + O(\sqrt{\lambda})$
- $P_R \varphi_n^R = P_R \left[\Psi_n^R + O(\sqrt{\lambda}) \right] = \Psi_n^R + O(\sqrt{\lambda}) = \varphi_n^R + O(\sqrt{\lambda})$
- $\hat{a}_R^+ \hat{a}_R \varphi_n^R = P_R a_R^+ a_R P_R \varphi_n^R = n \varphi_n^R + O(\sqrt{\lambda})$

thanks to the relation between φ_n^R and Ψ_n^R, or equivalently between $\mathcal{H}_R$ and $\mathcal{F}_R$.
Comments:

- Semiclassical state $\varphi_n^R \notin \mathcal{H}_R$, perturbative right Hilbert space
- $\varphi_n^R \in \mathcal{H}_R$, then $\Theta \varphi_n^R = \varphi_n^L \in \mathcal{H}_L$. But $\left(\varphi_n^R, \varphi_n^L \right) \neq 0$

While φ_n^R is not an element of the perturbative Hilbert space $\mathcal{H}_R$, its projection $P_R \varphi_n^R \in \mathcal{H}_R$ is non-perturbatively close to φ_n^R. In perturbation theory, one cannot distinguish the two states.

Perturbative states

Their support is concentrated around each minimum.
They are defined accordingly to their behavior as $\lambda \to 0$.

e.g. φ_n^R and φ_n^L implicitly depend on λ.

$\to$ We will refer to the elements of the family $\{\psi_\lambda\}_{\lambda>0}$ as a state $\psi_\lambda \in \mathcal{H}$.

def. ψ_λ is *perturbative with respect to the right minimum* if $F_R^{-1}\psi_\lambda$ converges in norm in $\mathcal{F}_R$ when $\lambda \to 0^+$.

Comments:

- all states φ_n^R are mapped to $|n\rangle_R \in \mathcal{F}_R$, which are λ-independent in $\mathcal{F}_R$. Hence all φ_n^R are trivially perturbative with respect to the right minimum.
- Ψ_0^+, which possesses two bumps. By going to $\mathcal{F}_R$ we may simply position ourselves at $x = x_R$ and send λ to zero. The right portion of the wave function then concentrates around the right minimum and approaches φ_0^R. As the left minimum moves away to $-\infty$, the left portion of the wave function is lost in the decoupling limit $\lambda = 0$.

$\to$ neither Ψ_0^+ nor Ψ_0^- are perturbative with respect to any minimum.

Perturbative operators

(i) preserve the direct sum up to non-perturbative effects

(ii) have a well-defined decoupling limit

$$A = \begin{pmatrix} A_{LL} & A_{LR} \\ A_{RL} & A_{RR} \end{pmatrix} \ : \ \mathcal{H}_L \oplus \mathcal{H}_R \to \mathcal{H}_L \oplus \mathcal{H}_R$$

then: (i) A_{LR} and A_{RL} must be non-perturbatively small

$$A_{LR} = o(\lambda^\infty) \text{ and } A_{RL} = o(\lambda^\infty);$$

and (ii) the decoupling limits $\lambda \to 0$ of A_{LL} in $\mathcal{H}_L$ and A_{RR} in $\mathcal{H}_R$ must exist.

Comments

- $\hat{a}_L, \hat{a}_L^+, \hat{a}_R, \hat{a}_R^+, \hat{N}_A$ by construction perturbative.
- In black hole physics, locality at the level of the effective bulk theory requires that left and right operators commute

$$[\hat{a}_L, \hat{a}_L^+] = [\hat{a}_R, \hat{a}_R^+] = 1 + o(\lambda^\infty), \qquad [\hat{a}_L, \hat{a}_R] = [\hat{a}_L, \hat{a}_R^+] = 0.$$

The canonical commutation relations on $\mathcal{H}_L$ and $\mathcal{H}_R$ are altered by a non-perturbative factor, while left and right operators commute.
[Papadodimas, Raju ‘13, Kabat, Lifshitz ‘14, Raju ‘16]

Perturbative operators - comments

- The original Hamiltonian H is a perturbative operator. The off-diagonal elements $P_R H P_L = o(\lambda^\infty)$ and $P_L H P_R = o(\lambda^\infty)$ are related to the tunneling rate and can be calculated by standard methods within the WKB approximation.
- This statement remains true for all energies, even if actual matrix elements become numerically large.

Operators on the tensor product

In the decoupling limit the structure of Hilbert space is expected to be that of a tensor product

$$\mathcal{F}_L \otimes \mathcal{F}_R \ , \qquad b_R, b_R^\dagger \ \ \rightarrow \ \ \mathbf{1} \otimes b_R \ , \ \ \mathbf{1} \otimes b_R^\dagger$$

Low energy states $\Psi_{mn} = \Psi_m^L + \Psi_n^R$ for $m, n \ll N$ approximate states of two decoupled harmonic oscillators.

$$A_{LL} \otimes \mathbf{1} + \mathbf{1} \otimes A_{RR} : \ \ \chi_m^L \otimes \chi_n^R \quad \rightarrow \quad (\lambda_m^L + \lambda_n^R)(\chi_m^L \otimes \chi_n^R).$$

Define *in the same way* on the direct sum, identified canonically

$$A(\chi_m^L, \chi_n^R) = (\lambda_m^L + \lambda_n^R)(\chi_m^L, \chi_n^R)$$

But the operator is non-linear. In order to preserve linearity:

$$A_R(\chi_m^L, \psi_R) = P_R A(\chi_m^L, \psi_R) = (\lambda_m^L \mathbf{1} + A_{RR})\psi_R.$$

which introduces state dependence.

$$\langle \chi_m^L \otimes \chi_n^R | A_\otimes | \chi_m^L \otimes \chi_n^R \rangle = \lambda_m^L + \lambda_n^R = \langle (\chi_m^L, \chi_n^R) | A | (\chi_m^L, \chi_n^R) \rangle.$$

Effective field theory states

Since $\mathcal{H}_L \oplus \mathcal{H}_R \cong \mathcal{H}_L \times \mathcal{H}_R$ through the canonical, bilinear map $s(\chi_L, \chi_R) = \chi_L \otimes \chi_R$ one can define states in the effective theory from perturbative states in $\mathcal{H}_L \otimes H_R$

$$\rho : \quad \chi = \chi_L + \chi_R \longmapsto \chi^{\text{eff}} = \mathcal{N} s(\chi) = \mathcal{N}\, \chi_L \otimes \chi_R,$$

If either $\chi_L = 0$ or $\chi_R = 0$, then $\chi^{\text{eff}} = 0$: if $\chi \in \mathcal{H}$ is perturbative with respect to any minimum, then $\chi^{\text{eff}} = 0$. We will say that a state is *typical* if it is represented by a non-vanishing effective state in the effective theory.

- The image of ρ is the set of $\psi_{eff} \in \mathcal{H}_{pert}$ with $\psi_{eff} \sim e^{i\theta}\psi_L \otimes \psi_R$. All states in $\mathcal{H}$ of the form

$$\psi = \alpha_L \psi_L + \alpha_R \psi_R, \text{ with } |\alpha_L|^2 + |\alpha_R|^2 = 1$$

- The state ψ^{eff} are *macrostates*, corresponding states ψ are *microstates*, thus parametrized by $(\alpha_L, \alpha_R) \in \mathbb{C}^2$

Non-perturbative effects allow to probe the wave function part, through the potential barrier. This is invisible in the effective theory, and its reflected on the appearance of macrostates.

Hawking radiation as tunneling

- WKB: $\Gamma = e^{-2\Lambda}$, $\Lambda = \int_{-x_1}^{x_1} \sqrt{2(V(x) - E)}dx$,
- $E \sim V_*$ $\quad \delta = V_* - E$.

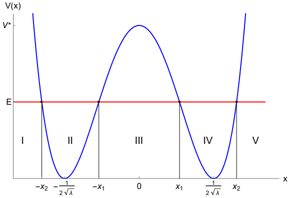
$$\Lambda(\lambda, \delta) = \sqrt{2}\pi\delta + 3\sqrt{2}\pi\lambda\delta^2 + O(\delta^3),$$

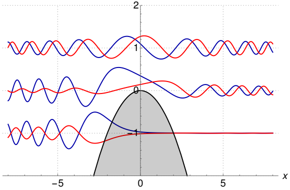
Approx. tunneling in an iho $V_{\text{iho}} = -\omega^2 \frac{x^2}{4}$, for a particle with $E = \omega\delta$.

- Black hole scattering matrix: $H_{\text{iho}} = \frac{p^2}{2} - \frac{1}{2}\nu^2 x^2$

$$T = \frac{e^{\pi\Delta/2\nu}}{\sqrt{2\pi}}\Gamma(\frac{1}{2} - \frac{i\Delta}{\nu}) , \qquad R = -i\frac{e^{-\pi\Delta/2\nu}}{\sqrt{2\pi}}\Gamma(\frac{1}{2} - \frac{i\Delta}{\nu}) ,$$

[Gaddam, Papadoulaki, Betzios '16]





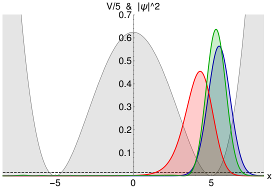
- Tunneling rate: $\Gamma_{\text{iho}}(E = \omega\delta) = \exp\left(-2\sqrt{2}\pi\delta - 6\sqrt{2}\pi\lambda\delta^2 + O(\delta^3)\right)$.

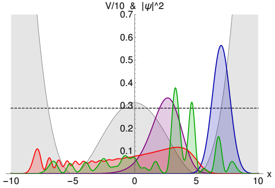
cft [Parhik, Wilczeck, '99], with $\omega \sim \sqrt{\lambda}\delta$, $M \sim 1/\sqrt{\lambda} = N$.

Chaotic evolution

Classical particle with energy $0 < E < V_* = V(0)$. Orbit remains 'outside the black hole', *i.e.*, it does not cross the maximum of the potential at $x = 0$.

- The period diverges logarithmically when $E \to V_*$:
  $T_{\text{trapped}} = \sqrt{2} \log\left(\frac{2}{\lambda\delta}\right) + O(\epsilon)$, sign of criticality
- Close to the tip, iho-driven dynamics:
  $x(t) = x_0 \cosh(\nu t) + v_0/\nu \sinh(\nu t)$
  at large times a position perturbation grows
  $\delta x(t) \sim e^{\nu t}(\delta x_0 + \delta v_0/\nu)$.
  This is by definition chaotic behavior with the Lapunov exponent
  $\nu = \omega/\sqrt{2} = 1/\sqrt{2}$.





Conclusions

- A quantum mechanics toy model where to study the details of perturbation theory.
- A naive perturbative limit gives rise to inconsistencies
- It is possible to give a well defined perturbative description, upon
  - Introducing degeneracy of states (entropy)
  - Introducing state dependence
- Perturbation theory breaks down when effects sensitive to the microstate structure take place

  [Ghosh, Raju ' 16]
- Dynamics of the model show characteristics of black hole physics

Thank You!